

ON COQUASITRIANGULAR BIALGEBRAS

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1. Introduction

Coquasitriangularity is one of the most fundamental concepts in quantum group theory. Some early papers on this notion are [LT], [S1], [H], [Doi], [Mj]. A bialgebra or a Hopf algebra is called *coquasitriangular* if it is equipped with a universal r -form. The latter concept is the main object of study in this paper.

Definition 1.1 Let \mathcal{A} be a bialgebra. A *universal r -form* on \mathcal{A} is a linear functional on $\mathcal{A} \otimes \mathcal{A}$ which is invertible with respect to the convolution multiplication and satisfies the following three conditions for arbitrary $a, b, c \in \mathcal{A}$:

- (CQT.1) $\mathbf{r}(c \otimes ab) = \mathbf{r}(c_{(1)} \otimes b)\mathbf{r}(c_{(2)} \otimes a),$
- (CQT.2) $\mathbf{r}(ab \otimes c) = \mathbf{r}(a \otimes c_{(1)})\mathbf{r}(b \otimes c_{(2)}),$
- (CQT.3) $\mathbf{r}(a_{(1)} \otimes b_{(1)})a_{(2)}b_{(2)} = \mathbf{r}(a_{(2)} \otimes b_{(2)})b_{(1)}a_{(1)}.$

The present paper deals with three topics on coquasitriangular bialgebras. In Section 2 we give a characterization of a universal r -form in terms of (certain) Yetter-Drinfeld modules. In Section 3 we study the uniqueness of universal r -forms for the coordinate Hopf algebras of the quantum groups $GL_q(N)$, $SL_q(N)$, $O_q(N)$ and $Sp_q(N)$. Section 4 is concerned with some linear functionals on a coquasitriangular Hopf algebra \mathcal{A} with universal r -form \mathbf{r} which describe the square of the antipode of \mathcal{A} . To be more precise, it is known that $S^2 = \bar{f}_{\mathbf{r}} * \text{id} * f_{\mathbf{r}}$, where $f_{\mathbf{r}}$ is the linear functional on \mathcal{A} defined by $f_{\mathbf{r}}(a) = \mathbf{r}(a_{(1)}, S(a_{(2)}))$, $a \in \mathcal{A}$, and $\bar{f}_{\mathbf{r}}$ is the convolution inverse of $f_{\mathbf{r}}$. Among others we show that $F_{\mathbf{r}} := f_{\mathbf{r}} * f_{\bar{\mathbf{r}}_{21}}$ is a character on \mathcal{A} which coincides with Woronowicz's modular character f_{-2} when \mathcal{A} is the coordinate Hopf $*$ -Algebra for the standard compact quantum groups $U_q(N)$, $SU_q(N)$, $O_q(N; \mathbb{R})$, and $USp_q(N)$. Dual versions for quasitriangular Hopf algebras of some results in Section 4 have been proved by V.G. Drinfeld [Dr].

Let us fix some notation. We use the Sweedler notations $\Delta(a) = a_{(1)} \otimes a_{(2)}$ for the comultiplication Δ and $\delta(m) = m_{(0)} \otimes m_{(1)}$ for a right coaction δ . The multiplication map of an algebra \mathcal{A} is denoted by $m_{\mathcal{A}}$. We write $*$ for the convolution multiplication. The convolution inverse of a functional f is denoted by \bar{f} .

2. Universal \mathbf{r} -forms and Yetter-Drinfeld modules

If not stated otherwise, \mathcal{A} denotes a bialgebra in this section. First we recall a well-known definition (see [Y], [S2] or [Mo]).

Definition 2.1. A (right) Yetter-Drinfeld module of \mathcal{A} is a right \mathcal{A} -module and a right \mathcal{A} -comodule M satisfying the compatibility condition

$$m_{(0)} \triangleleft a_{(1)} \otimes m_{(1)} a_{(2)} = (m \triangleleft a_{(2)})_{(0)} \otimes a_{(1)} (m \triangleleft a_{(2)})_{(1)} \quad (1)$$

for all $m \in M$ and $a \in \mathcal{A}$. Here the right action of $a \in \mathcal{A}$ on $m \in M$ is denoted by $m \triangleleft a$ and the right coaction $\delta: M \rightarrow M \otimes \mathcal{A}$ is expressed by the Sweedler notation $\delta(m) = m_{(0)} \otimes m_{(1)}, m \in M$.

For a right \mathcal{A} -comodule M , let $\mathcal{C}(M) = \text{Lin}\{m'(m_{(0)}) m_{(1)} | m \in M, m' \in M'\}$ denote the coefficient coalgebra of M (see, for instance, [KS], p.399).

For the following considerations we assume that \mathbf{r} is a *convolution invertible* linear functional on the bialgebra $\mathcal{A} \otimes \mathcal{A}$. Recall that the convolution inverse of \mathbf{r} is denoted by $\bar{\mathbf{r}}$.

Let M be a right \mathcal{A} -comodule with right coaction $\delta(m) = m_{(0)} \otimes m_{(1)}$. For $m \in M$ and $a \in \mathcal{A}$, we define

$$m \triangleleft_1 a = \mathbf{r}(m_{(1)} \otimes a) m_{(0)} \quad \text{and} \quad m \triangleleft_2 a = \bar{\mathbf{r}}(a \otimes m_{(1)}) m_{(0)}.$$

Let $M_i, i = 1, 2$, denote the right \mathcal{A} -comodule M equipped with the mapping $M \times \mathcal{A} \ni (m, a) \rightarrow m \triangleleft_i a \in M$. If \mathbf{r} is a universal r -form, then it is known (see [Mo], Example 10.6.14) that M_1 is a Yetter-Drinfeld module. We shall strengthen this fact and give a characterization of universal r -forms in this manner.

Lemma 2.2 (i) M_1 is a Yetter-Drinfeld module of \mathcal{A} with right action \triangleleft_1 if and only if (CQT.1) holds for all $a, b \in \mathcal{A}$ and $c \in \mathcal{C}(M)$ and (CQT.3) holds for all $a \in \mathcal{C}(M)$ and $b \in \mathcal{A}$.

(ii) M_2 is a Yetter-Drinfeld module of \mathcal{A} with right action \triangleleft_2 if and only if (CQT.2) is fulfilled for all $a, b \in \mathcal{A}$ and $c \in \mathcal{C}(M)$ and (CQT.3) holds for all $a \in \mathcal{A}$ and $b \in \mathcal{C}(M)$.

Proof. We prove the assertion for M_1 . It is obvious that the condition $(m \triangleleft_1 a) \triangleleft_1 b = m \triangleleft_1 ab$ is equivalent to equation (CQT.1) for $c \in \mathcal{C}(M)$ and $a, b \in \mathcal{A}$. We show that the latter implies that $m \triangleleft 1 = m, m \in M$. Indeed, using the convolution inverse $\bar{\mathbf{r}}$ of \mathbf{r} and condition (CQT.1) for $a = b = 1$, we get

$$\bar{\mathbf{r}}(c \otimes 1) = \bar{\mathbf{r}}(c_{(1)} \otimes 1) \mathbf{r}(c_{(2)} \otimes 1) \mathbf{r}(c_{(3)} \otimes 1) = \bar{\mathbf{r}}(c_{(1)} \otimes 1) \mathbf{r}(c_{(2)} \otimes 1) = \varepsilon(c).$$

for $c \in \mathcal{C}(M)$, so that $m \triangleleft 1 = \mathbf{r}(m_{(1)} \otimes 1) m_{(0)} = m$. The left hand and right hand sides of the Yetter-Drinfeld condition (1) are equal to

$$\mathbf{r}(m_{(1)} \otimes a_{(1)}) m_{(0)} \otimes m_{(2)} a_{(2)} \quad \text{and} \quad \mathbf{r}(m_{(2)} \otimes a_{(2)}) m_{(0)} \otimes a_{(1)} m_{(1)},$$

respectively. Thus, (1) is equivalent to (CQT.3) for $a \in \mathcal{C}(M)$ and $b \in \mathcal{A}$.

The assertion for M_2 follows by some slight modifications of the preceding reasoning. The

relation $(m \triangleleft_2 a) \triangleleft_2 b = m \triangleleft_2 ab$, where $a, b \in \mathcal{A}$ and $m \in M$, is fulfilled iff $\bar{\mathbf{r}}(a \otimes c_{(2)})\bar{\mathbf{r}}(b \otimes c_{(1)}) = \bar{\mathbf{r}}(ab \otimes c)$ for $a, b \in \mathcal{A}$ and $c \in \mathcal{C}(M)$. The latter is obviously equivalent to the fact that $\mathbf{r}(a \otimes c_{(1)})\mathbf{r}(b \otimes c_{(2)}) = \mathbf{r}(ab \otimes c)$ for $a, b \in \mathcal{A}$ and $c \in \mathcal{A}$. The Yetter-Drinfeld condition (1) is satisfied iff $\bar{\mathbf{r}}(a_{(1)} \otimes b_{(1)})b_{(2)}a_{(2)} = \bar{\mathbf{r}}(a_{(2)} \otimes b_{(2)})a_{(1)}b_{(1)}$ for all $a \in \mathcal{A}$ and $b \in \mathcal{C}(M)$ which in turn is equivalent to equation (CQT.3) for $a \in \mathcal{A}$ and $b \in \mathcal{C}(M)$. ■

An immediate consequence of Lemma 2.2 is

Proposition 2.3 *Let \mathcal{R} be a family of right comodules of a bialgebra \mathcal{A} such that the linear span of the coefficient coalgebras $\mathcal{C}(M)$, $M \in \mathcal{R}$, coincides with \mathcal{A} . Let \mathbf{r} be a convolution invertible linear functional on $\mathcal{A} \otimes \mathcal{A}$. Then \mathbf{r} is a universal r -form of \mathcal{A} if and only if M_1 and M_2 are Yetter-Drinfeld modules for any comodule $M \in \mathcal{R}$.*

Corollary 2.4 *A convolution invertible linear functional \mathbf{r} on $\mathcal{A} \otimes \mathcal{A}$ is a universal r -form of a bialgebra \mathcal{A} if and only if \mathcal{A} becomes a Yetter-Drinfeld module with the comultiplication as right coaction and with the mappings \triangleleft_1 and \triangleleft_2 as right actions.*

Remark 2.5 As shown by P. Schauenburg [S2], the category of right Yetter-Drinfeld modules is equivalent to the category of Hopf bimodules over a Hopf algebra \mathcal{A} . Using this correspondence the preceding results can be reformulated in terms of Hopf bimodules of \mathcal{A} . In this setting they were crucial for the construction of bicovariant differential calculi on general coquasitriangular Hopf algebras (see [SS], last line on p.189, and [KS], Section 14.5).

3. Uniqueness of universal r -forms for quantized matrix groups

In this section let G_q denote one of the quantum groups $GL_q(N)$, $SL_q(N)$, $O_q(N)$ or $Sp_q(N)$ and $\mathcal{O}(G_q)$ its coordinate Hopf algebra as defined in [FRT], [T1] or [KS], Chapter 9. It is known (see, for instance, [KS], Theorem 10.9) that the Hopf algebra $\mathcal{O}(G_q)$ is coquasitriangular and there exists a universal r -form \mathbf{r}_z of $\mathcal{O}(G_q)$ such that

$$\mathbf{r}_z(u_j^i \otimes u_m^n) = z R_{jm}^{in}, \quad i, j, n, m = 1, \dots, N. \quad (2)$$

Here $u = (u_j^i)_{i,j=1,\dots,N}$ is the fundamental matrix of $\mathcal{O}(G_q)$, R is the corresponding R -matrix as given in [FRT], (1.5) and (1.9), or in [KS], (9.13) and (9.30), and z is a fixed complex number such that $z \neq 0$ for $G_q = GL_q(N)$, $z^N = q^{-1}$ for $G_q = SL_q(N)$ and $z^2 = 1$ for $G_q = O_q(N), Sp_q(N)$.

Throughout this section we assume that q is a complex number which is not a root of unity. (A closer look at the proof given below shows that it suffices to exclude only very few roots of unity.) We shall show that the above functionals \mathbf{r}_z exhaust *all* universal r -forms of $\mathcal{O}(G_q)$. In order to place this result in a more general context, we need some preliminaries.

Definition 3.1 A *central bicharacter* of a bialgebra \mathcal{A} is a convolution invertible linear functional \mathbf{c} on $\mathcal{A} \otimes \mathcal{A}$ such that for arbitrary $a, b, c \in \mathcal{A}$:

$$(CB.1) \quad \mathbf{c}(ab \otimes c) = \mathbf{c}(a \otimes c_{(1)})\mathbf{c}(b \otimes c_{(2)}) \text{ and } \mathbf{c}(c \otimes ab) = \mathbf{c}(c_{(1)} \otimes b)\mathbf{c}(c_{(2)} \otimes a),$$

$$(CB.2) \quad \mathbf{c}(a \otimes b_{(1)})b_{(2)} = \mathbf{c}(a \otimes b_{(2)})b_{(1)} \text{ and } \mathbf{c}(a_{(1)} \otimes b)a_{(2)} = \mathbf{c}(a_{(2)} \otimes b)a_{(1)}.$$

Condition (CB.1) means that $\mathbf{c}(\cdot \otimes \cdot)$ is a dual pairing of the bialgebras \mathcal{A} and \mathcal{A}^{op} , where \mathcal{A}^{op} is the bialgebra with the same comultiplication and the opposite multiplication as \mathcal{A} . Condition (CB.2) is equivalent to the requirement that for any $a \in \mathcal{A}$ the linear functionals $\mathbf{c}(a \otimes \cdot)$ and $\mathbf{c}(\cdot \otimes a)$ on \mathcal{A} are central in the dual algebra \mathcal{A}' . A trivial example of central bicharacter is the counit of $\mathcal{A} \otimes \mathcal{A}$.

Suppose that \mathbf{c} is a central bicharacter and \mathbf{r} is a universal r -form of \mathcal{A} . Using the condition (CQT.1)–(CQT.3) and (CB.1)–(CB.2) it is a straightforward matter to verify that the convolution product $\mathbf{c} * \mathbf{r}$ is again a universal r -form of \mathcal{A} . Recall ([KS], Proposition 10.2(iv)) that $\bar{\mathbf{r}}_{21}$ is another universal r -form of \mathcal{A} , where $\bar{\mathbf{r}}_{21}(a \otimes b) := \bar{\mathbf{r}}(b \otimes a)$, $a, b \in \mathcal{A}$. Thus $\mathbf{c} * \bar{\mathbf{r}}_{21}$ is also a universal r -form of \mathcal{A} .

Definition 3.2 A coquasitriangular bialgebra \mathcal{A} is said to have an *essentially unique universal r -form* if there exists a universal r -form \mathbf{r} of \mathcal{A} such that for any universal r -form \mathbf{s} of \mathcal{A} there exists a central bicharacter \mathbf{c} such that $\mathbf{s} = \mathbf{c} * \mathbf{r}$ or $\mathbf{s} = \mathbf{c} * \bar{\mathbf{r}}_{21}$.

Example 3.3 Let $\mathcal{A} = \mathbb{C}G$ be the group Hopf algebra of an abelian group G . Then the universal r -forms of \mathcal{A} are in one-to-one correspondence to the bicharacters of the group G . Since $\mathbb{C}G$ is cocommutative and G is abelian, the universal r -forms are precisely the central bicharacters of \mathcal{A} and hence $\mathcal{A} = \mathbb{C}G$ has obviously an essentially unique universal r -form.

Let us return to the coquasitriangular Hopf algebra $\mathcal{O}(G_q)$. We shall say that a complex number ζ is *admissible* if $\zeta \neq 0$ for $G_q = GL_q(N)$, $\zeta^N = 1$ for $G_q = SL_q(N)$ and $\zeta^2 = 1$ for $G_q = O_q(N), Sp_q(N)$. For any admissible number ζ there exists a unique central bicharacter \mathbf{c}_ζ of $\mathcal{O}(G_q)$ such that

$$\mathbf{c}_\zeta(u_j^i \otimes u_m^n) = \delta_{ij} \delta_{nm} \zeta, \quad i, j, n, m = 1, \dots, N. \quad (3)$$

In order to prove this, one first extends (3) to a linear functional on $\mathbb{C}\langle u_j^i \rangle \otimes \mathbb{C}\langle u_j^i \rangle$ such that (CB.1) is satisfied, where $\mathbb{C}\langle u_j^i \rangle$ denotes the free bialgebra with generators u_j^i , $i, j = 1, \dots, N$. Because ζ is assumed to be admissible, it follows from the defining relations for the algebra $\mathcal{O}(G_q)$ that this functional passes to a functional \mathbf{c}_ζ of $\mathcal{O}(G_q) \otimes \mathcal{O}(G_q)$. It is easily seen that \mathbf{c}_ζ is a central bicharacter of $\mathcal{O}(G_q)$.

We now fix a universal r -form \mathbf{r}_{z_0} of $\mathcal{O}(G_q)$ with parameter z_0 as above and denote it by \mathbf{r} . Then it is clear that any universal r -form \mathbf{r}_z of $\mathcal{O}(G_q)$ given by (2) is of the form $\mathbf{r}_z = \mathbf{c}_\zeta * \mathbf{r}$ for some admissible number ζ .

Proposition 3.4 *Suppose that q is not a root of unity. For any universal r -form \mathbf{s} of $\mathcal{O}(G_q)$ there exists an admissible complex number ζ such that $\mathbf{s} = \mathbf{c}_\zeta * \mathbf{r}$ or $\mathbf{s} = \mathbf{c}_\zeta * \bar{\mathbf{r}}_{21}$. The coquasitriangular Hopf algebra $\mathcal{O}(G_q)$ has an essentially unique universal r -form.*

Proof. Suppose that \mathbf{s} is a universal r -form of $\mathcal{O}(G_q)$. We define $N^2 \times N^2$ -matrices $T = (T_{jm}^{in})$ and $\hat{T} = (\hat{T}_{jm}^{ni})$ by

$$\hat{T}_{jm}^{ni} = T_{jm}^{in} := s(u_j^i \otimes u_m^n), \quad i, j, n, m = 1, \dots, N.$$

From the general theory of coquasitriangular Hopf algebras (see, for instance, [KS], 10.1) we conclude that \hat{T} belongs to the centralizer algebra $\text{Mor}(u \otimes u)$ of the tensor product

corepresentation $u \otimes u$ and that T satisfies the quantum Yang-Baxter equation, so \hat{T} fulfills the braid relation

$$\hat{T}_{12}\hat{T}_{23}\hat{T}_{12} = \hat{T}_{23}\hat{T}_{12}\hat{T}_{23} \quad (4)$$

on the space of the tensor product corepresentation $u \otimes u \otimes u$.

We shall carry out the proof in the cases $O_q(N)$ and $Sp_q(N)$. Then it is well-known (see [Re] or [KS], Proposition 8.40) that there exists a homomorphism π of the Birman-Wenzl-Murakami algebra $\text{BWM}_3(q, \epsilon q^{N-\epsilon})$ to the centralizer algebra $\text{Mor}(u \otimes u \otimes u)$ such that $\pi(G_i) = \hat{R}_{i,i+1}$, $i = 1, 2$, where $\epsilon = 1$ for $O_q(N)$ and $\epsilon = -1$ for $Sp_q(N)$. The generators of the BWM-algebra $\text{BWM}_3(q, \epsilon q^{N-\epsilon})$ are denoted by G_1, G_2, E_1, E_2 and their images under the homomorphism π by g_1, g_2, e_1, e_2 . The matrices $\{\hat{R}, \hat{R}^{-1}, I\}$ form a basis of the vector space $\text{Mor}(u \otimes u)$. Since $\hat{T} \in \text{Mor}(u \otimes u)$, there are complex numbers α, β, γ such that $\hat{T} = \alpha\hat{R} + \beta\hat{R}^{-1} + \gamma \cdot I$. Thus, we have

$$\hat{T}_{i,i+1} = \alpha g_i + \beta g_i^{-1} + \gamma \cdot I. \quad (5)$$

The crucial step of the proof is to show that the braid relation for \hat{T} implies that \hat{T} is a complex multiple of either \hat{R}, \hat{R}^{-1} or I . In order to prove this, we essentially use the relations of the BWM-algebra $\text{BWM}_3(q, \epsilon q^{N-\epsilon})$ (see [BW], p. 225).

Inserting (5) into (4), both sides of (4) are sums of 27 summands. From the braid relation $G_1G_2G_1 = G_2G_1G_2$ in the BWM-algebra we get $g_1g_2g_1 = g_2g_1g_2$, $g_1^{-1}g_2^{-1}g_1^{-1} = g_2^{-1}g_1^{-1}g_2^{-1}$, $g_1^{-1}g_2g_1 = g_2g_1g_2^{-1}$, $g_1g_2^{-1}g_1 = g_2^{-1}g_1^{-1}g_2$, $g_1^{-1}g_2^{-1}g_1 = g_2g_1^{-1}g_2^{-1}$ and $g_1g_2g_1^{-1} = g_2^{-1}g_1g_2$. Inserting these relations and cancelling equal terms on both sides of (4) we finally obtain

$$\begin{aligned} & \alpha^2\beta g_1g_2^{-1}g_1 + \alpha\beta^2g_1^{-1}g_2g_1^{-1} + \alpha^2\gamma g_1^2 + \beta^2\gamma g_1^{-2} \\ &= \alpha^2\beta g_2g_1^{-1}g_2 + \alpha\beta^2g_2^{-1}g_1g_2^{-1} + \alpha^2\gamma g_2^2 + \beta^2\gamma g_2^{-2}. \end{aligned} \quad (6)$$

Now we recall the following relations in the BWM-algebra (see [BW], p. 255):

$$\begin{aligned} G_i^{-1} - G_i &= \lambda E_i - \lambda \cdot 1, G_2E_1G_2 = G_1^{-1}E_2G_1^{-1}, G_2^{-1}E_1G_2^{-1} = G_1E_2G_1, \\ E_1E_2G_1 &= E_1G_2^{-1}, G_1E_2E_1 = G_2^{-1}E_1, E_1E_2E_1 = E_1, \end{aligned}$$

where $\lambda := q - q^{-1}$. Applying the images of these relations under the homomorphism π , a straightforward computation shows that (6) reduces to the equation

$$\begin{aligned} & \alpha^2(\gamma - \beta\lambda)(g_1^2 - g_2^2) + \beta^2(\alpha\lambda + \gamma)(g_1^{-2} - g_2^{-2}) \\ &+ \alpha\beta(\alpha + \beta)\lambda^2\{e_2g_1 + g_1e_2 - e_1g_2^{-1} - g_2^{-1}e_1 + \lambda(e_1e_2 + e_2e_1 + e_1 + e_2)\} = 0. \end{aligned} \quad (7)$$

The BWM-algebra $\text{BWM}_3(q, \epsilon q^{N-\epsilon})$ is 15-dimensional and the elements

$$1, G_1, G_2, G_1G_2, G_2G_1, G_1G_2G_1, E_1, E_2, E_1E_2, E_2E_1, G_1E_2, E_2G_1, G_2^{-1}E_1, E_1G_2^{-1}, G_1E_2G_1 \quad (8)$$

form a vector space basis (see [BW] or [KS]), Proposition 8.39). From the representation theory of quantum groups it is well-known (see, for instance, [KS], 8.6.2) how to decompose

the tensor product $u \otimes u \otimes u$ into irreducible components. From these decompositions it follows that the homomorphism π is injective for $O_q(N)$, $N \geq 3$, and for $Sp_q(N)$, $N \geq 6$. In these cases the images of the elements (8) under π are also linearly independent. Therefore, the coefficient of, say, $e_2 g_1$ in (7) is zero, so that

$$\alpha\beta(\alpha + \beta) = 0. \quad (9)$$

Hence the second line of (7) vanishes identically and the coefficients of g_1 and e_1 in (7) have to be zero as well. Using the relation $G_1 E_1 = \epsilon q^{\epsilon-N} E_1$ in the algebra $BWM_3(q, \epsilon q^{N-\epsilon})$, these coefficients are computed as

$$-\alpha^2(\gamma\beta\lambda)\lambda + \beta^2(\alpha\lambda + \gamma)\lambda = 0, \quad (10)$$

$$\alpha^2(\gamma - \beta\lambda)\lambda\epsilon q^{\epsilon-N} + \beta^2(\alpha\lambda + \gamma)\lambda(\epsilon q^{N-\epsilon} - \lambda) = 0, \quad (11)$$

respectively. In the case of the quantum group $Sp_q(4)$ the corepresentation corresponding to the Young tableaux of a column with 3 boxes does not occur in the decomposition of tensor product $u \otimes u \otimes u$ (see [KS], p.289) and we have $\dim \text{Mor}(u \otimes u \otimes u) = 14$. An explicit computation shows that the images of the basis elements (8) satisfy the linear relation

$$\begin{aligned} &1 - q^{-1}g_1 - q^{-1}g_2 + g^{-2}g_1g_2 + q^{-2}g_2g_1 - q^{-3}g_1g_2g_1 - q^{-6}e_1 - q^{-2}e_1 - q^{-4}e_1e_2 \\ &- q^{-4}e_2e_1 + q^{-3}g_1e_2 + q^{-3}e_2g_1 + g^{-5}e_1g_2^{-1} + q^{-5}g_2^{-1}e_1 - q^{-4}g_1e_2g_1 = 0. \end{aligned} \quad (12)$$

Therefore, the derivation of equations (9)–(11) from (7) is also valid in the case $Sp_q(4)$. Since $(q - \epsilon q^{\epsilon-N})(q + \epsilon q^{\epsilon-N}) \neq 0$ by assumption, (10) and (11) imply that

$$\alpha^2(\gamma - \beta\lambda) = \beta^2(\alpha\lambda + \gamma) = 0. \quad (13)$$

The solutions of equations (9) and (13) are $\beta = \gamma = 0, \alpha = \gamma = 0$ and $\alpha = \beta = 0$ which means that \hat{T} is a multiple of either \hat{R} , \hat{R}^{-1} or I .

For the quantum groups $GL_q(N)$ and $SL_q(N)$ the proof is similar and much simpler. Then there is a homomorphism of the Hecke algebra $H_3(q)$ on the centralizer algebra $\text{Mor}(u \otimes u \otimes u)$ which is injective for $N \geq 3$. In the case $N=2$ we have $\dim H_3(q) = 1 + \dim \text{Mor}(u \otimes u \otimes u) = 6$ and the corresponding linear relation is obtained from (12) by setting $e_1 = e_2 = 0$ therein. Since the Hopf algebras $\mathcal{O}(Sp_q(2))$ and $\mathcal{O}(SL_{q^2}(2))$ are isomorphic, we also cover the case $Sp_q(2)$ in this manner which was excluded during preceding considerations.

Summarizing, we have shown that $\hat{T} = z \cdot \hat{R}$ or $\hat{T} = z \cdot \hat{R}^{-1}$ or $\hat{T} = z \cdot I$ for some complex number z . The rest of the proof is more or less routine. Using the fact that \mathbf{s} is a dual pairing of $\mathcal{O}(G_q)$ and $\mathcal{O}(G_q)^{\text{op}}$ it follows that the case $\hat{T} = z \cdot I$ is impossible (because it is not compatible with the defining relation $\hat{R}_1 u_1 u_2 = u_1 u_2 \hat{R}$) and that the number z must be as described at the beginning of this section. Thus we have $\mathbf{s} = \mathbf{r}_z$ or $\mathbf{s} = (\bar{\mathbf{r}}_z)_{21}$. Fixing a universal r -form \mathbf{r}_{z_0} and reformulating the latter in terms of a central bicharacter \mathbf{c}_ζ , the proof will be completed. \blacksquare

Remark 3.5 As I have learned from the referee, the universal r -forms of the bialgebra $\mathcal{O}(M_q(N))$ have been described recently in the paper [T2]. This result implies the assertion of Proposition 3.4 in the case $G_q = GL_q(N)$.

4. On functionals describing the square of the antipode

Throughout this section we assume that \mathcal{A} is a coquasitriangular Hopf algebra and \mathbf{r} is a universal r -form of \mathcal{A} .

Let $f_{\mathbf{r}}$ and $\bar{f}_{\mathbf{r}}$ denote the linear functionals on \mathcal{A} defined by

$$f_{\mathbf{r}}(a) = \mathbf{r}(a_{(1)} \otimes S(a_{(2)})) \text{ and } \bar{f}_{\mathbf{r}}(a) = \bar{\mathbf{r}}(S(a_{(1)}) \otimes a_{(2)}), \quad a \in \mathcal{A}. \quad (14)$$

Then it is well-known (see, for instance, [KS], Proposition 10.3) that $\bar{f}_{\mathbf{r}}$ is the convolution inverse of $f_{\mathbf{r}}$ and that the square of the antipode S is given by

$$S^2 = \bar{f}_{\mathbf{r}} * \text{id} * f_{\mathbf{r}}, \quad \text{that is, } S^2(a) = \bar{f}_{\mathbf{r}}(a_{(1)})a_{(2)}f_{\mathbf{r}}(a_{(3)}), \quad a \in \mathcal{A}. \quad (15)$$

Lemma 4.1 *The functional $f_{\mathbf{r}}$ satisfies the equations*

$$\mathbf{r}_{21} * \mathbf{r} * (f_{\mathbf{r}} \circ m_{\mathcal{A}}) = (f_{\mathbf{r}} \circ m_{\mathcal{A}}) * \mathbf{r}_{21} * \mathbf{r} = f_{\mathbf{r}} \otimes f_{\mathbf{r}}. \quad (16)$$

Proof. Using the properties (CQT.1)–(CQT.3) of the universal r -form \mathbf{r} we compute

$$\begin{aligned} & \mathbf{r}_{21}(a_{(1)} \otimes b_{(1)})\mathbf{r}(a_{(2)} \otimes b_{(2)})f_{\mathbf{r}}(a_{(3)}b_{(3)}) \\ &= \mathbf{r}(b_{(1)} \otimes a_{(1)})\mathbf{r}(a_{(2)} \otimes b_{(2)})\mathbf{r}(a_{(3)}b_{(3)} \otimes S(a_{(4)}b_{(4)})) \\ &= \mathbf{r}(b_{(1)} \otimes a_{(1)})\mathbf{r}(a_{(3)} \otimes b_{(3)})\mathbf{r}(b_{(2)}a_{(2)} \otimes S(b_{(4)})S(a_{(4)})) \\ &= \mathbf{r}(a_{(4)} \otimes b_{(4)})\mathbf{r}(b_{(1)} \otimes a_{(1)})\mathbf{r}(b_{(2)}a_{(2)} \otimes S(a_{(5)}))\mathbf{r}(b_{(3)}a_{(3)} \otimes S(b_{(5)})) \\ &= \mathbf{r}(a_{(4)} \otimes b_{(4)})\mathbf{r}(b_{(2)} \otimes a_{(2)})\mathbf{r}(a_{(1)}b_{(1)} \otimes S(a_{(5)}))\mathbf{r}(b_{(3)} \otimes S(b_{(6)}))\mathbf{r}(a_{(3)} \otimes S(b_{(5)})) \\ &= \mathbf{r}(b_{(2)} \otimes a_{(2)})\mathbf{r}(a_{(1)}b_{(1)} \otimes S(a_{(3)}))\mathbf{r}(b_{(3)} \otimes S(b_{(4)})) \\ &= \mathbf{r}(b_{(2)} \otimes a_{(2)})\mathbf{r}(a_{(1)} \otimes S(a_{(4)}))\mathbf{r}(b_{(1)} \otimes S(a_{(3)}))f_{\mathbf{r}}(b_{(3)}) \\ &= f_{\mathbf{r}}(a)f_{\mathbf{r}}(b) \end{aligned}$$

for $a, b \in \mathcal{A}$. This proves the first equality $\mathbf{r}_{21} * \mathbf{r} * (f_{\mathbf{r}} \circ m_{\mathcal{A}}) = f_{\mathbf{r}} \otimes f_{\mathbf{r}}$. Applying condition (CQT.3) twice to this relation we get the second equality $(f_{\mathbf{r}} \circ m_{\mathcal{A}}) * \mathbf{r}_{21} * \mathbf{r} = f_{\mathbf{r}} \otimes f_{\mathbf{r}}$. \blacksquare

In some sense the functional $\mathbf{r}_{21} * \mathbf{r}$ on $\mathcal{A} \otimes \mathcal{A}$ measures the distance of $f_{\mathbf{r}}$ from being a character. In particular, Lemma 3.1 implies

Corollary 4.2. *The functional $f_{\mathbf{r}}$ is a character of \mathcal{A} (that is, $f_{\mathbf{r}}(ab) = f_{\mathbf{r}}(a)f_{\mathbf{r}}(b)$ for $a, b \in \mathcal{A}$) if and only if \mathcal{A} is cotriangular (that is, $\bar{\mathbf{r}} = \mathbf{r}_{21}$).*

Recall that any coquasitriangular Hopf algebra \mathcal{A} has a second universal r -form $\mathbf{s} := \bar{\mathbf{r}}_{21}$ given by $\mathbf{s}(a \otimes b) = \bar{\mathbf{r}}(b \otimes a)$, $a, b \in \mathcal{A}$.

Proposition 4.3 *The functionals $f_{\mathbf{r}}, \bar{f}_{\mathbf{r}}, f_{\mathbf{s}}, \bar{f}_{\mathbf{s}}$ pairwise commute in the algebra \mathcal{A}° , $z := f_{\mathbf{r}} * \bar{f}_{\mathbf{s}}$ belongs to the center of \mathcal{A}' and $g := f_{\mathbf{r}} * f_{\mathbf{s}}$ is a character of \mathcal{A} such that $S^4 = \bar{g} * \text{id} * g$.*

Proof. The antipode S is bijective and we have $S * \bar{f}_s = \bar{f}_s * S^{-1}$ ([KS], Proposition 10.3). Using this fact, the relation $\mathbf{r}(S(a) \otimes S(b)) = \mathbf{r}(a \otimes b)$ and (3), we obtain

$$\begin{aligned} f_{\mathbf{r}} * \bar{f}_s(a) &= \mathbf{r}(a_{(1)} \otimes S(a_{(2)})) \bar{f}_s(a_{(3)}) = \mathbf{r}(a_{(1)} \otimes S^{-1}(a_{(3)})) \bar{f}_s(a_{(2)}) \\ &= \mathbf{r}(S^2(a_{(1)}) \otimes S(a_{(3)})) \bar{f}_s(a_{(2)}) = \mathbf{r}(a_{(2)} \otimes S(a_{(3)})) \bar{f}_s(a_{(1)}) = \bar{f}_s * f_{\mathbf{r}}(a). \end{aligned}$$

Hence the functionals $f_{\mathbf{r}}, \bar{f}_{\mathbf{r}}, f_{\mathbf{s}}, \bar{f}_{\mathbf{s}}$ pairwise commute.

Applying (3) to both \mathbf{r} and \mathbf{s} , we get

$$z * \text{id} = f_{\mathbf{r}} * \bar{f}_{\mathbf{s}} * \text{id} = f_{\mathbf{r}} * S^2 * \bar{f}_{\mathbf{s}} = \text{id} * f_{\mathbf{r}} * \bar{f}_{\mathbf{s}} = \text{id} * z,$$

so z is in the center of the algebra \mathcal{A}' .

Since $\mathbf{r}_{21} * \mathbf{r} * \mathbf{s}_{21} * \mathbf{s} = \varepsilon_{\mathcal{A} \otimes \mathcal{A}}$, the equations (16) easily imply that $g(ab) = g(a)g(b)$ for $a, b \in \mathcal{A}$, that is, g is a character on \mathcal{A} . By (15), we have $S^4 = \bar{g} * \text{id} * g$. ■

We illustrate the preceding by a simple example.

Example 4.4 Let \mathcal{A} be the Hopf algebra $\mathbb{C}\mathbb{Z}$ of the group of integers. Then any universal r -form of \mathcal{A} is of the form $\mathbf{r}(n \otimes m) = \lambda^{nm}$, $n, m \in \mathbb{Z}$, for some fixed $\lambda \in \mathbb{C}, \lambda \neq 0$ (see also Example 3.3). Thus we have $f_{\mathbf{r}}(n) = \lambda^{n^2}$, $f_{\mathbf{s}}(n) = \lambda^{-n^2}$ and $F_{\mathbf{r}}(n) = 1$ for $n \in \mathbb{Z}$, so that $F_{\mathbf{r}} = \varepsilon$. Since $f_{\mathbf{r}}(n+m) = f_{\mathbf{r}}(n)f_{\mathbf{r}}(m)\lambda^{2nm}$ for $n, m \in \mathbb{Z}$, the functional $f_{\mathbf{r}}$ on \mathcal{A} is far from being a character in general.

Let us suppose now that \mathcal{A} is a coquasitriangular Hopf algebra equipped with a universal r -form \mathbf{r} and that \mathcal{A} is also a CQG -algebra (see [DK] or [KS], 11.3.1, for this notion). Let $f_z, z \in \mathbb{C}$, denote Woronowicz's modular characters on \mathcal{A} (see [W] or [KS], 11.3.4). Recall that $f_z * f_{z'} = f_{z+z'}$, for $z, z' \in \mathbb{C}$ and $S^2(a) = f_2 * \text{id} * f_{-2}$ for $a \in \mathcal{A}$. Then the functionals $F_{\mathbf{r}}$ and f_{-2} are both characters on \mathcal{A} which implement S^4 , that is,

$$S^4(a) = \bar{F}_{\mathbf{r}}(a_{(1)})a_{(2)}F_{\mathbf{r}}(a_{(3)}) = f_2(a_{(1)})a_{(2)}f_{-2}(a_{(3)}), \quad a \in \mathcal{A}.$$

Hence $F_{\mathbf{r}}f_2$ is a character of \mathcal{A} which is central in \mathcal{A}' . This suggests the following

PROBLEM: *Do the characters $F_{\mathbf{r}}$ and f_{-2} on \mathcal{A} coincide?*

If the Hopf algebra \mathcal{A} is cocommutative (in particular, if \mathcal{A} is the group algebra $\mathbb{C}G$ of an abelian group G), then we have $F_{\mathbf{r}} = f_{-2} = \varepsilon$ and so the answer is affirmative.

A more interesting case is the Hopf $*$ -algebra $\mathcal{O}(G_q)$, where G_q is one of the compact forms $U_q(N)$, $SU_q(N)$, $O_q(N, \mathbb{R})$, $USp_q(N)$ of the quantum groups $GL_q(N)$, $SL_q(N)$, $O_q(N)$, $Sp_q(N)$, respectively, and q is real. Then $\mathcal{O}(G_q)$ is a CQG -algebra ([KS], Example 11.7) and a coquasitriangular Hopf algebra with universal r -form \mathbf{r} determined by (2).

Proposition 4.5 *Then we have $F_{\mathbf{r}} = f_{-2}$.*

Proof. From the explicit formulas for $f_{\mathbf{r}}(u_j^i), f_{\mathbf{s}}(u_j^i)$ and $f_{-2}(u_j^i)$ listed in [KS], p.341 and p.425, respectively, we see that $F_{\mathbf{r}}(u_j^i) = f_{-2}(u_j^i)$ for $i, j = 1, \dots, N$. Therefore, since $F_{\mathbf{r}}$ and f_{-2} are both characters, they coincide on the whole algebra $\mathcal{O}(G_q)$. ■

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